Name of college - S.S. college, J Bad

Deptt - Mathematics

Topic - Problem on ~~Rolle's~~ Mean value theorem (Real Analysis)

Class - B.Sc part-II

Time - 7.30 A.M to 9.A.M

Date - ~~28~~ 27-04-2021

Teacher's Name-

Dr. Shree Nath Sharma

(Associate prof)

Problems based on Mean Value theorem

Problem: Verify first mean value theorem in the interval (0, 4) for the function

$$f(x) = (x-1)(x-2)(x-3)$$

Solution: Here $f(x) = (x-1)(x-2)(x-3)$

$= x^3 - 6x^2 + 11x - 6$

$\Rightarrow f'(x) = 3x^2 - 12x + 11$

$\Rightarrow f'(c) = 3c^2 - 12c + 11$ ——— (1)

Let $a = 0$ $b = 4$

$f(0) \Rightarrow f(a) = -6$ $f(4) = f(b) = 6$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{6+6}{4-0}$$

$= 3$

But $f'(c) = 3c^2 - 12c + 11$

$\Rightarrow 3 = 3c^2 - 12c + 11$

$\Rightarrow 3c^2 - 12c + 8 = 0$

Since it is a quadratic equation in c

$$\therefore c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$= \frac{12 \pm \sqrt{48}}{6}$$

$$= \frac{12 \pm 4\sqrt{3}}{6} = \frac{\cancel{12}^{2}}{\cancel{6}} \pm \frac{\cancel{4}^{2}\cancel{\sqrt{3}}}{\cancel{6}_{3}\sqrt{3}}$$

$$= 2 \pm \frac{2}{\sqrt{3}}$$

$\therefore c = 2 + \frac{2}{\sqrt{3}}$ or $c = 2 - \frac{2}{\sqrt{3}}$ both these values of c lie in (0, 4) and hence the mean value theorem is verified for $f(x)$ in (0, 4)

Problem

Examine the validity of the hypothesis and conclusion of Lagrange's Mean Value theorem for the function f defined by $f(x) = |x|$ for every $x \in [-2,1]$

Solution

Here. $f(x) = |x| \quad \forall \; x \in [-2,1]$

Here $f(x) = |x|$ is continuous over $[-2,1]$ but $f(x)$ is not differentiable over $[-2,1]$

As $f(x)$ is not differentiable at 0

As

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - 0}{h} =$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1$$

Here $Rf'(0)$ and $Lf'(0)$ exist and finite

but $Rf'(0) \neq Lf'(0)$

$\Rightarrow f(x)$ is not differentiable at $x = 0$

Thus the hypothesis of Lagrange's Mean theorem is not satisfied.

If there exists $c \in [-2,1]$

Such that $\dfrac{f(1) - f(-2)}{1-(-2)} = f'(c)$

$$\Rightarrow \frac{1-2}{1+2} = f'(c) \Rightarrow f'(c) = -1/3$$

Hence $f(x) = -x/3 + k$ at $x = c$

Problem :→ Verify Lagrange's mean value theorem for $f(x) = \log x$ in $(1, e)$

Solution :→ Here $f(x) = \log x \Rightarrow f'(x) = 1/x$

$\Rightarrow f'(c) = \frac{1}{c}$

Here $a = 1 \Rightarrow f(a) = f(1) = \log 1 = 0$

$b = e \Rightarrow f(b) = \log_e e = 1$

From Lagrange's Mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \frac{1}{c} = \frac{1 - 0}{e - 1}$$

$\Rightarrow (e - 1) = c$

$\Rightarrow c = e - 1$ which lies in $(1, e)$

$\because 2 < e < 3$

and hence Lagrange's Mean theorem is verified in the interval $(1, e)$ for the given function.

Problem:- A function $f(x)$ is defined by

$f(x) = x \sin \frac{1}{x}$ in $(-1, 1)$

$f(0) = 0$

Are the conditions of first Mean theorem satisfied in this case.

Soln :→ Given that $f(x) = x \sin \frac{1}{x}$ in $(-1, 1)$

Also $f(0) = 0$

Also $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

$\therefore$ we find that $\lim_{x \to 0} f(x) = f(0) \Rightarrow$ the function is continuous at $x = 0$

Again let us consider at $x = a$

where $-1 < a < 1$

and $\lim_{x \to a} f(x) = \lim_{x \to a} x \sin(\frac{1}{x})$

$= a \sin \frac{1}{a}$

$\therefore \lim_{x \to a} f(x) = f(a)$ and finite

$\Rightarrow$ the function is continuous at $x = a$

where: $-1 \le a \le 1$

thus the function $f(x)$ is continuous in the closed interval $[-1, 1]$

But $f(x) = x \sin\frac{1}{x}$ is not differentiable at $x = 0$ which lies in the interval $(-1, 1)$

Hence the condition of first mean theorem are not satisfied and therefore it can not be applied to $f(x)$ in the interval $(-1, 1)$

Problem verify First Mean value theorem in $[0, \frac{1}{2}]$ for the function $f(x) = x(x-1)(x-2)$

Solution :→ By the question

$f(x) = x(x-1)(x-2)$

$f(0) = 0 \quad f(\frac{1}{2}) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2}) = \frac{3}{8}$

clearly $f(x)$ is continuous over $[0, \frac{1}{2}]$ and differentiable over $(0, \frac{1}{2})$

/ Being the polynomial function, it is differentiable and also continuous in said interval /

thus the hypothesis of Lagrange's Mean value theorem is satisfied. In order to verify that the conclusion also holds. we must find a point $c \in (0, \frac{1}{2})$ such that

$$\frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = f'(c)$$

Now $f(x) = x(x-1)(x-2)$

$\Rightarrow \quad = x(x^2 - 3x + 2)$

$= x^3 - 3x^2 + 2x$

$\Rightarrow f'(x) = 3x^2 - 6x + 2 \Rightarrow f'(c) = 3c^2 - 6c + 2$

$\therefore \frac{3/8}{1/2} = \frac{3}{4} = 3c^2 - 6c + 2 \Rightarrow 12c^2 - 24c + 8 - 3 = 0$

$12c^2 - 24c + 5 = 0$

$\therefore c = 1 \pm \frac{1}{6}\sqrt{21}$ when $-$ve sign is taken, $c = 1 - \frac{1}{6}\sqrt{21} < \frac{1}{2}$ $\therefore c = 1 - \sqrt{\frac{7}{12}}$